数値解析　（塩田）

—— 数 値 微 分 ——

中心差分

$$\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{1}{2h}\{y(x+h)-y(x-h)\} \\
\frac{\delta^2 y}{\delta x^2} &= \frac{1}{h^2}\{y(x+h)-2y(x)+y(x-h)\} \\
\frac{\delta^3 y}{\delta x^3} &= \frac{1}{2h^3}\{y(x+2h)-2y(x+h)+2y(x-h)-y(x-2h)\} \\
\frac{\delta^4 y}{\delta x^4} &= \frac{1}{h^4}\{y(x+2h)-4y(x+h)+6y(x)-4y(x-h)+y(x-2h)\} \\
\frac{\delta^5 y}{\delta x^5} &= \frac{1}{2h^5}\{y(x+3h)-4y(x+2h)+5y(x+h)-5y(x-h) \\
&\qquad +4y(x-2h)-y(x-3h)\} \\
\frac{\delta^6 y}{\delta x^6} &= \frac{1}{h^6}\{y(x+3h)-6y(x+2h)+15y(x+h)-20y(x)+15y(x-h) \\
&\qquad -6y(x-2h)+y(x-3h)\} \\
\frac{\delta^7 y}{\delta x^7} &= \frac{1}{2h^7}\{y(x+4h)-6y(x+3h)+14y(x+2h)-14y(x+h) \\
&\qquad +14y(x-h)-14y(x-2h)+6y(x-3h)-y(x-4h)\} \\
\frac{\delta^8 y}{\delta x^8} &= \frac{1}{h^8}\{y(x+4h)-8y(x+3h)+28y(x+2h)-56y(x+h)+70y(x) \\
&\qquad -56y(x-h)+28y(x-2h)-8y(x-3h)+y(x-4h)\}
\end{aligned}$$

中心差分を用いた補正公式

$$\begin{aligned}
y'(x) &= \frac{\delta y}{\delta x}-\frac{1}{6}\frac{\delta^3 y}{\delta x^3}h^2+\frac{1}{30}\frac{\delta^5 y}{\delta x^5}h^4-\frac{1}{140}\frac{\delta^7 y}{\delta x^7}h^6+\cdots \\
y''(x) &= \frac{\delta^2 y}{\delta x^2}-\frac{1}{12}\frac{\delta^4 y}{\delta x^4}h^2+\frac{1}{90}\frac{\delta^6 y}{\delta x^6}h^4-\frac{1}{560}\frac{\delta^8 y}{\delta x^8}h^6+\cdots \\
y^{(3)}(x) &= \frac{\delta^3 y}{\delta x^3}-\frac{1}{4}\frac{\delta^5 y}{\delta x^5}h^2+\frac{7}{120}\frac{\delta^7 y}{\delta x^7}h^4-\cdots \\
y^{(4)}(x) &= \frac{\delta^4 y}{\delta x^4}-\frac{1}{6}\frac{\delta^6 y}{\delta x^6}h^2+\frac{7}{240}\frac{\delta^8 y}{\delta x^8}h^4-\cdots
\end{aligned}$$

前進差分

$$\begin{aligned}
\frac{\varDelta y}{\varDelta x} &= \frac{1}{h}\{y(x+h)-y(x)\} \\
\frac{\varDelta^2 y}{\varDelta x^2} &= \frac{1}{h^2}\{y(x+2h)-2y(x+h)+y(x)\} \\
\frac{\varDelta^3 y}{\varDelta x^3} &= \frac{1}{h^3}\{y(x+3h)-3y(x+2h)+3y(x+h)-y(x)\}
\end{aligned}$$

$$
\begin{aligned}
\frac{\varDelta^4 y}{\varDelta x^4} &= \frac{1}{h^4}\{y(x+4h) - 4y(x+3h) + 6y(x+2h) - 4y(x+h) + y(x)\} \\
\frac{\varDelta^5 y}{\varDelta x^5} &= \frac{1}{h^5}\{y(x+5h) - 5y(x+4h) + 10y(x+3h) - 10y(x+2h) \\
&\qquad +5y(x+h) - y(x)\} \\
\frac{\varDelta^k y}{\varDelta x^k} &= \frac{1}{h^k}\sum_{j=0}^{k}(-1)^{k-j}\left(\begin{array}{c} k \\ j \end{array}\right) y(x+jh) \qquad \left(\text{ただし} \left(\begin{array}{c} k \\ j \end{array}\right) \text{は二項係数} \right)
\end{aligned}
$$

前進差分を用いた補正公式

$$
\begin{aligned}
y'(x) &= \frac{\varDelta y}{\varDelta x} - \frac{1}{2}\frac{\varDelta^2 y}{\varDelta x^2}h + \frac{1}{3}\frac{\varDelta^3 y}{\varDelta x^3}h^2 - \frac{1}{4}\frac{\varDelta^4 y}{\varDelta x^4}h^3 + \frac{1}{5}\frac{\varDelta^5 y}{\varDelta x^5}h^4 - \frac{1}{6}\frac{\varDelta^6 y}{\varDelta x^6}h^5 \\
&\quad +\frac{1}{7}\frac{\varDelta^7 y}{\varDelta x^7}h^6 - \frac{1}{8}\frac{\varDelta^8 y}{\varDelta x^8}h^7 + \cdots \\
y''(x) &= \frac{\varDelta^2 y}{\varDelta x^2} - \frac{\varDelta^3 y}{\varDelta x^3}h + \frac{11}{12}\frac{\varDelta^4 y}{\varDelta x^4}h^2 - \frac{5}{6}\frac{\varDelta^5 y}{\varDelta x^5}h^3 + \frac{137}{180}\frac{\varDelta^6 y}{\varDelta x^6}h^4 - \frac{7}{10}\frac{\varDelta^7 y}{\varDelta x^7}h^5 \\
&\quad +\frac{363}{560}\frac{\varDelta^8 y}{\varDelta x^8}h^6 - \cdots \\
y^{(3)}(x) &= \frac{\varDelta^3 y}{\varDelta x^3} - \frac{3}{2}\frac{\varDelta^4 y}{\varDelta x^4}h + \frac{7}{4}\frac{\varDelta^5 y}{\varDelta x^5}h^2 - \frac{15}{8}\frac{\varDelta^6 y}{\varDelta x^6}h^3 + \frac{29}{15}\frac{\varDelta^7 y}{\varDelta x^7}h^4 \\
&\quad -\frac{469}{240}\frac{\varDelta^8 y}{\varDelta x^8}h^5 + \cdots \\
y^{(4)}(x) &= \frac{\varDelta^4 y}{\varDelta x^4} - 2\frac{\varDelta^5 y}{\varDelta x^5}h + \frac{17}{6}\frac{\varDelta^6 y}{\varDelta x^6}h^2 - \frac{7}{2}\frac{\varDelta^7 y}{\varDelta x^7}h^3 + \frac{967}{240}\frac{\varDelta^8 y}{\varDelta x^8}h^4 - \cdots
\end{aligned}
$$

ラプラシアンフィルタ

2 変数関数 $u(x,y)$ の x による偏微分は y を定数と思って計算するので、例えば 2 階偏微分は中心差分を用いて

$$
\frac{\partial^2 u}{\partial x^2} \qquad \frac{1}{h^2}\{u(x+h,y) - 2u(x,y) + u(x-h,y)\}
$$

と近似できる。同様に

$$
\frac{\partial^2 u}{\partial y^2} \qquad \frac{1}{h^2}\{u(x,y+h) - 2u(x,y) + u(x,y-h)\}
$$

物理学でよく使うラプラシアンの数値微分はこの 2 式を足せばよく、

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad & \frac{1}{h^2}\{u(x+h,y) + u(x-h,y) \\
& \qquad + u(x,y+h) + u(x,y-h) - 4u(x,y)\}
\end{aligned}
$$

となる。この係数を絵で表せば

$y+h$		1	
y	1	-4	1
$y-h$		1	
	$x-h$	x	$x+h$

これは画像処理ではラプラシアンフィルタと呼ばれ、輪郭抽出などに用いられる。