数値解析　（塩田）

—— 数 値 微 分 ——

中心差分

$$
\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{1}{2h}\{y(x+h)-y(x-h)\} \\
\frac{\delta^2 y}{\delta x^2} &= \frac{1}{h^2}\{y(x+h)-2y(x)+y(x-h)\} \\
\frac{\delta^3 y}{\delta x^3} &= \frac{1}{2h^3}\{y(x+2h)-2y(x+h)+2y(x-h)-y(x-2h)\} \\
\frac{\delta^4 y}{\delta x^4} &= \frac{1}{h^4}\{y(x+2h)-4y(x+h)+6y(x)-4y(x-h)+y(x-2h)\} \\
\frac{\delta^5 y}{\delta x^5} &= \frac{1}{2h^5}\{y(x+3h)-4y(x+2h)+5y(x+h)-5y(x-h) \\
&\qquad +4y(x-2h)-y(x-3h)\} \\
\frac{\delta^6 y}{\delta x^6} &= \frac{1}{h^6}\{y(x+3h)-6y(x+2h)+15y(x+h)-20y(x)+15y(x-h) \\
&\qquad -6y(x-2h)+y(x-3h)\} \\
\frac{\delta^7 y}{\delta x^7} &= \frac{1}{2h^7}\{y(x+4h)-6y(x+3h)+14y(x+2h)-14y(x+h) \\
&\qquad +14y(x-h)-14y(x-2h)+6y(x-3h)-y(x-4h)\} \\
\frac{\delta^8 y}{\delta x^8} &= \frac{1}{h^8}\{y(x+4h)-8y(x+3h)+28y(x+2h)-56y(x+h)+70y(x) \\
&\qquad -56y(x-h)+28y(x-2h)-8y(x-3h)+y(x-4h)\}
\end{aligned}
$$

中心差分を用いた補正公式

$$
\begin{aligned}
y'(x) &= \frac{\delta y}{\delta x}-\frac{1}{6}\frac{\delta^3 y}{\delta x^3}h^2+\frac{1}{30}\frac{\delta^5 y}{\delta x^5}h^4-\frac{1}{140}\frac{\delta^7 y}{\delta x^7}h^6+\cdots \\
y''(x) &= \frac{\delta^2 y}{\delta x^2}-\frac{1}{12}\frac{\delta^4 y}{\delta x^4}h^2+\frac{1}{90}\frac{\delta^6 y}{\delta x^6}h^4-\frac{1}{560}\frac{\delta^8 y}{\delta x^8}h^6+\cdots \\
y^{(3)}(x) &= \frac{\delta^3 y}{\delta x^3}-\frac{1}{4}\frac{\delta^5 y}{\delta x^5}h^2+\frac{7}{120}\frac{\delta^7 y}{\delta x^7}h^4-\cdots \\
y^{(4)}(x) &= \frac{\delta^4 y}{\delta x^4}-\frac{1}{6}\frac{\delta^6 y}{\delta x^6}h^2+\frac{7}{240}\frac{\delta^8 y}{\delta x^8}h^4-\cdots
\end{aligned}
$$

前進差分

$$
\begin{aligned}
\frac{\varDelta y}{\varDelta x} &= \frac{1}{h}\{y(x+h)-y(x)\} \\
\frac{\varDelta^2 y}{\varDelta x^2} &= \frac{1}{h^2}\{y(x+2h)-2y(x+h)+y(x)\} \\
\frac{\varDelta^3 y}{\varDelta x^3} &= \frac{1}{h^3}\{y(x+3h)-3y(x+2h)+3y(x+h)-y(x)\} \\
\frac{\varDelta^4 y}{\varDelta x^4} &= \frac{1}{h^4}\{y(x+4h)-4y(x+3h)+6y(x+2h)-4y(x+h)+y(x)\} \\
\frac{\varDelta^5 y}{\varDelta x^5} &= \frac{1}{h^5}\{y(x+5h)-5y(x+4h)+10y(x+3h)-10y(x+2h) \\
&\qquad +5y(x+h)-y(x)\} \\
\frac{\varDelta^6 y}{\varDelta x^6} &= \frac{1}{h^6}\{y(x+6h)-6y(x+5h)+15y(x+4h)-20y(x+3h) \\
&\qquad +15y(x+2h)-6y(x+h)+y(x)\} \\
\frac{\varDelta^7 y}{\varDelta x^7} &= \frac{1}{h^7}\{y(x+7h)-7y(x+6h)+21y(x+5h)-35y(x+4h) \\
&\qquad +35y(x+3h)-21y(x+2h)+7y(x+h)-y(x)\} \\
\frac{\varDelta^8 y}{\varDelta x^8} &= \frac{1}{h^8}\{y(x+8h)-8y(x+7h)+28y(x+6h)-56y(x+5h) \\
&\qquad +70y(x+4h)-56y(x+3h)+28y(x+2h)-8y(x+h)+y(x)\}
\end{aligned}
$$

前進差分を用いた補正公式

$$
\begin{aligned}
y'(x) &= \frac{\varDelta y}{\varDelta x}-\frac{1}{2}\frac{\varDelta^2 y}{\varDelta x^2}h+\frac{1}{3}\frac{\varDelta^3 y}{\varDelta x^3}h^2-\frac{1}{4}\frac{\varDelta^4 y}{\varDelta x^4}h^3+\frac{1}{5}\frac{\varDelta^5 y}{\varDelta x^5}h^4-\frac{1}{6}\frac{\varDelta^6 y}{\varDelta x^6}h^5 \\
&\qquad +\frac{1}{7}\frac{\varDelta^7 y}{\varDelta x^7}h^6-\frac{1}{8}\frac{\varDelta^8 y}{\varDelta x^8}h^7+\cdots \\
y''(x) &= \frac{\varDelta^2 y}{\varDelta x^2}-\frac{\varDelta^3 y}{\varDelta x^3}h+\frac{11}{12}\frac{\varDelta^4 y}{\varDelta x^4}h^2-\frac{5}{6}\frac{\varDelta^5 y}{\varDelta x^5}h^3+\frac{137}{180}\frac{\varDelta^6 y}{\varDelta x^6}h^4-\frac{7}{10}\frac{\varDelta^7 y}{\varDelta x^7}h^5 \\
&\qquad +\frac{363}{560}\frac{\varDelta^8 y}{\varDelta x^8}h^6-\cdots \\
y^{(3)}(x) &= \frac{\varDelta^3 y}{\varDelta x^3}-\frac{3}{2}\frac{\varDelta^4 y}{\varDelta x^4}h+\frac{7}{4}\frac{\varDelta^5 y}{\varDelta x^5}h^2-\frac{15}{8}\frac{\varDelta^6 y}{\varDelta x^6}h^3+\frac{29}{15}\frac{\varDelta^7 y}{\varDelta x^7}h^4 \\
&\qquad -\frac{469}{240}\frac{\varDelta^8 y}{\varDelta x^8}h^5+\cdots \\
y^{(4)}(x) &= \frac{\varDelta^4 y}{\varDelta x^4}-2\frac{\varDelta^5 y}{\varDelta x^5}h+\frac{17}{6}\frac{\varDelta^6 y}{\varDelta x^6}h^2-\frac{7}{2}\frac{\varDelta^7 y}{\varDelta x^7}h^3+\frac{967}{240}\frac{\varDelta^8 y}{\varDelta x^8}h^4-\cdots
\end{aligned}
$$

ラプラシアンフィルタ

2 変数関数 $u(x, y)$ の x による偏微分は y を定数と思って計算するので、例えば 2 階偏微分は中心差分を用いて

$$\frac{\partial^2 u}{\partial x^2} \quad \frac{1}{h^2}\{u(x+h, y) - 2u(x, y) + u(x-h, y)\}$$

と近似できる。同様に

$$\frac{\partial^2 u}{\partial y^2} \quad \frac{1}{h^2}\{u(x, y+h) - 2u(x, y) + u(x, y-h)\}$$

物理学でよく使うラプラシアンの数値微分はこの 2 式を足せばよく、

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \frac{1}{h^2}\{u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)\}$$

となる。この係数を絵で表せば

$y+h$		1	
y	1	-4	1
$y-h$		1	
	$x-h$	x	$x+h$

これは画像処理ではラプラシアンフィルタと呼ばれ、輪郭抽出などに用いられる。