

数值解析 (塩田)

— 数值微分 —

中心差分

$$\frac{\delta y}{\delta x} = \frac{1}{2h} \{y(x+h) - y(x-h)\}$$

$$\frac{\delta^2 y}{\delta x^2} = \frac{1}{h^2} \{y(x+h) - 2y(x) + y(x-h)\}$$

$$\frac{\delta^3 y}{\delta x^3} = \frac{1}{2h^3} \{y(x+2h) - 2y(x+h) + 2y(x-h) - y(x-2h)\}$$

$$\frac{\delta^4 y}{\delta x^4} = \frac{1}{h^4} \{y(x+2h) - 4y(x+h) + 6y(x) - 4y(x-h) + y(x-2h)\}$$

$$\frac{\delta^5 y}{\delta x^5} = \frac{1}{2h^5} \{y(x+3h) - 4y(x+2h) + 5y(x+h) - 5y(x-h) + 4y(x-2h) - y(x-3h)\}$$

$$\frac{\delta^6 y}{\delta x^6} = \frac{1}{h^6} \{y(x+3h) - 6y(x+2h) + 15y(x+h) - 20y(x) + 15y(x-h) - 6y(x-2h) + y(x-3h)\}$$

$$\frac{\delta^7 y}{\delta x^7} = \frac{1}{2h^7} \{y(x+4h) - 6y(x+3h) + 14y(x+2h) - 14y(x+h) + 14y(x-h) - 14y(x-2h) + 6y(x-3h) - y(x-4h)\}$$

$$\frac{\delta^8 y}{\delta x^8} = \frac{1}{h^8} \{y(x+4h) - 8y(x+3h) + 28y(x+2h) - 56y(x+h) + 70y(x) - 56y(x-h) + 28y(x-2h) - 8y(x-3h) + y(x-4h)\}$$

中心差分による補正公式

$$y'(x) = \frac{\delta y}{\delta x} - \frac{1}{6} \frac{\delta^3 y}{\delta x^3} h^2 + \frac{1}{30} \frac{\delta^5 y}{\delta x^5} h^4 - \frac{1}{140} \frac{\delta^7 y}{\delta x^7} h^6 + \dots$$

$$y''(x) = \frac{\delta^2 y}{\delta x^2} - \frac{1}{12} \frac{\delta^4 y}{\delta x^4} h^2 + \frac{1}{90} \frac{\delta^6 y}{\delta x^6} h^4 - \frac{1}{560} \frac{\delta^8 y}{\delta x^8} h^6 + \dots$$

$$y'''(x) = \frac{\delta^3 y}{\delta x^3} - \frac{1}{4} \frac{\delta^5 y}{\delta x^5} h^2 + \frac{7}{120} \frac{\delta^7 y}{\delta x^7} h^4 - \dots$$

$$y''''(x) = \frac{\delta^4 y}{\delta x^4} - \frac{1}{6} \frac{\delta^6 y}{\delta x^6} h^2 + \frac{7}{240} \frac{\delta^8 y}{\delta x^8} h^4 - \dots$$

前進差分

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{1}{h} \{y(x+h) - y(x)\} \\ \frac{\Delta^2 y}{\Delta x^2} &= \frac{1}{h^2} \{y(x+2h) - 2y(x+h) + y(x)\} \\ \frac{\Delta^3 y}{\Delta x^3} &= \frac{1}{h^3} \{y(x+3h) - 3y(x+2h) + 3y(x+h) - y(x)\} \\ \frac{\Delta^4 y}{\Delta x^4} &= \frac{1}{h^4} \{y(x+4h) - 4y(x+3h) + 6y(x+2h) - 4y(x+h) + y(x)\} \\ \frac{\Delta^5 y}{\Delta x^5} &= \frac{1}{h^5} \{y(x+5h) - 5y(x+4h) + 10y(x+3h) - 10y(x+2h) \\ &\quad + 5y(x+h) - y(x)\} \\ \frac{\Delta^6 y}{\Delta x^6} &= \frac{1}{h^6} \{y(x+6h) - 6y(x+5h) + 15y(x+4h) - 20y(x+3h) \\ &\quad + 15y(x+2h) - 6y(x+h) + y(x)\} \\ \frac{\Delta^7 y}{\Delta x^7} &= \frac{1}{h^7} \{y(x+7h) - 7y(x+6h) + 21y(x+5h) - 35y(x+4h) \\ &\quad + 35y(x+3h) - 21y(x+2h) + 7y(x+h) - y(x)\} \\ \frac{\Delta^8 y}{\Delta x^8} &= \frac{1}{h^8} \{y(x+8h) - 8y(x+7h) + 28y(x+6h) - 56y(x+5h) \\ &\quad + 70y(x+4h) - 56y(x+3h) + 28y(x+2h) - 8y(x+h) + y(x)\} \end{aligned}$$

前進差分による補正公式

$$\begin{aligned} y'(x) &= \frac{\Delta y}{\Delta x} - \frac{1}{2} \frac{\Delta^2 y}{\Delta x^2} h + \frac{1}{3} \frac{\Delta^3 y}{\Delta x^3} h^2 - \frac{1}{4} \frac{\Delta^4 y}{\Delta x^4} h^3 + \frac{1}{5} \frac{\Delta^5 y}{\Delta x^5} h^4 - \frac{1}{6} \frac{\Delta^6 y}{\Delta x^6} h^5 \\ &\quad + \frac{1}{7} \frac{\Delta^7 y}{\Delta x^7} h^6 - \frac{1}{8} \frac{\Delta^8 y}{\Delta x^8} h^7 + \dots \\ y''(x) &= \frac{\Delta^2 y}{\Delta x^2} - \frac{\Delta^3 y}{\Delta x^3} h + \frac{11}{12} \frac{\Delta^4 y}{\Delta x^4} h^2 - \frac{5}{6} \frac{\Delta^5 y}{\Delta x^5} h^3 + \frac{137}{180} \frac{\Delta^6 y}{\Delta x^6} h^4 - \frac{7}{10} \frac{\Delta^7 y}{\Delta x^7} h^5 \\ &\quad + \frac{363}{560} \frac{\Delta^8 y}{\Delta x^8} h^6 - \dots \\ y'''(x) &= \frac{\Delta^3 y}{\Delta x^3} - \frac{3}{2} \frac{\Delta^4 y}{\Delta x^4} h + \frac{7}{4} \frac{\Delta^5 y}{\Delta x^5} h^2 - \frac{15}{8} \frac{\Delta^6 y}{\Delta x^6} h^3 + \frac{29}{15} \frac{\Delta^7 y}{\Delta x^7} h^4 \\ &\quad - \frac{469}{240} \frac{\Delta^8 y}{\Delta x^8} h^5 + \dots \\ y''''(x) &= \frac{\Delta^4 y}{\Delta x^4} - 2 \frac{\Delta^5 y}{\Delta x^5} h + \frac{17}{6} \frac{\Delta^6 y}{\Delta x^6} h^2 - \frac{7}{2} \frac{\Delta^7 y}{\Delta x^7} h^3 + \frac{967}{240} \frac{\Delta^8 y}{\Delta x^8} h^4 - \dots \end{aligned}$$

中心差分による補正公式を検証する Mathematica プログラム

```
(*
h = the step width
y0 = y(x), y1 = 1st derivative of y(x), y2 = 2nd derivative of y(x), ...
      d1 = 1st central difference, d2 = 2nd central difference, ...
*)

Clear[y0,y1,y2,y3,y4,y5,y6,y7,y8,d1,d2,d3,d4,d5,d6,d7,d8,h];

y[h_]:=y0+y1*h+y2*h^2/2+y3*h^3/6+y4*h^4/24+y5*h^5/120+y6*h^6/720+y7*h^7/5040+y8*h^8/40320;
(* Taylor expansion of y(x+h) *)

main[]:=Module[{},
  d1=Expand[(y[h]-y[-h])/(2*h)];
  d2=Expand[(y[h]-2*y[0]+y[-h])/(h^2)];
  d3=Expand[(y[2*h]-2*y[h]+2*y[-h]-y[-2*h])/(2*h^3)];
  d4=Expand[(y[2*h]-4*y[h]+6*y[0]-4*y[-h]+y[-2*h])/(h^4)];
  d5=Expand[(y[3*h]-4*y[2*h]+5*y[h]-5*y[-h]+4*y[-2*h]-y[-3*h])/(2*h^5)];
  d6=Expand[(y[3*h]-6*y[2*h]+15*y[h]-20*y[0]+15*y[-h]-6*y[-2*h]+y[-3*h])/(h^6)];
  d7=Expand[(y[4*h]-6*y[3*h]+14*y[2*h]-14*y[h]+14*y[-h]-14*y[-2*h]+6*y[-3*h]-y[-4*h])
    /(2*h^7)];
  d8=Expand[(y[4*h]-8*y[3*h]+28*y[2*h]-56*y[h]+70*y[0]-56*y[-h]+28*y[-2*h]-8*y[-3*h]
    +y[-4*h])/(h^8)];

  Print[];
  Print["d1 = ",InputForm[d1]];
  Print["d2 = ",InputForm[d2]];
  Print["d3 = ",InputForm[d3]];
  Print["d4 = ",InputForm[d4]];
  Print["d5 = ",InputForm[d5]];
  Print["d6 = ",InputForm[d6]];
  Print["d7 = ",InputForm[d7]];
  Print["d8 = ",InputForm[d8]];
  Print["-----"];
  z=Expand[d1-d3*h^2/6+d5*h^4/30-d7*h^6/140];
  Print["d1-d3*h^2/6+d5*h^4/30-d7*h^6/140 = ",InputForm[z]];
  z=Expand[d2-d4*h^2/12+d6*h^4/90-d8*h^6/560];
  Print["d2-d4*h^2/12+d6*h^4/90-d8*h^6/560 = ",InputForm[z]];
  z=Expand[d3-d5*h^2/4+(7/120)*d7*h^4];
  Print["d3-d5*h^2/4+(7/120)*d7*h^4 = ",InputForm[z]];
  z=Expand[d4-d6*h^2/6+(7/240)*d8*h^4];
  Print["d4-d6*h^2/6+(7/240)*d8*h^4 = ",InputForm[z]];
];

main[];

(*
d1 = y1 + (h^2*y3)/6 + (h^4*y5)/120 + (h^6*y7)/5040
d2 = y2 + (h^2*y4)/12 + (h^4*y6)/360 + (h^6*y8)/20160
d3 = y3 + (h^2*y5)/4 + (h^4*y7)/40
d4 = y4 + (h^2*y6)/6 + (h^4*y8)/80
d5 = y5 + (h^2*y7)/3
d6 = y6 + (h^2*y8)/4
d7 = y7
d8 = y8
-----
d1-d3*h^2/6+d5*h^4/30-d7*h^6/140 = y1
d2-d4*h^2/12+d6*h^4/90-d8*h^6/560 = y2
d3-d5*h^2/4+(7/120)*d7*h^4 = y3
d4-d6*h^2/6+(7/240)*d8*h^4 = y4
*)
```