数値解析 (数値解析 C)　(塩田)

—— 数 値 微 分 ——

中心差分

$$
\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{1}{2h}\{y(x+h)-y(x-h)\} \\
\frac{\delta^2 y}{\delta x^2} &= \frac{1}{h^2}\{y(x+h)-2y(x)+y(x-h)\} \\
\frac{\delta^3 y}{\delta x^3} &= \frac{1}{2h^3}\{y(x+2h)-2y(x+h)+2y(x-h)-y(x-2h)\} \\
\frac{\delta^4 y}{\delta x^4} &= \frac{1}{h^4}\{y(x+2h)-4y(x+h)+6y(x)-4y(x-h)+y(x-2h)\} \\
\frac{\delta^5 y}{\delta x^5} &= \frac{1}{2h^5}\{y(x+3h)-4y(x+2h)+5y(x+h)-5y(x-h) \\
&\qquad +4y(x-2h)-y(x-3h)\} \\
\frac{\delta^6 y}{\delta x^6} &= \frac{1}{h^6}\{y(x+3h)-6y(x+2h)+15y(x+h)-20y(x)+15y(x-h) \\
&\qquad -6y(x-2h)+y(x-3h)\} \\
\frac{\delta^7 y}{\delta x^7} &= \frac{1}{2h^7}\{y(x+4h)-6y(x+3h)+14y(x+2h)-14y(x+h) \\
&\qquad +14y(x-h)-14y(x-2h)+6y(x-3h)-y(x-4h)\} \\
\frac{\delta^8 y}{\delta x^8} &= \frac{1}{h^8}\{y(x+4h)-8y(x+3h)+28y(x+2h)-56y(x+h)+70y(x) \\
&\qquad -56y(x-h)+28y(x-2h)-8y(x-3h)+y(x-4h)\}
\end{aligned}
$$

中心差分による補正公式

$$
\begin{aligned}
y'(x) &= \frac{\delta y}{\delta x}-\frac{1}{6}\frac{\delta^3 y}{\delta x^3}h^2+\frac{1}{30}\frac{\delta^5 y}{\delta x^5}h^4-\frac{1}{140}\frac{\delta^7 y}{\delta x^7}h^6+\cdots \\
y''(x) &= \frac{\delta^2 y}{\delta x^2}-\frac{1}{12}\frac{\delta^4 y}{\delta x^4}h^2+\frac{1}{90}\frac{\delta^6 y}{\delta x^6}h^4-\frac{1}{560}\frac{\delta^8 y}{\delta x^8}h^6+\cdots \\
y'''(x) &= \frac{\delta^3 y}{\delta x^3}-\frac{1}{4}\frac{\delta^5 y}{\delta x^5}h^2+\frac{7}{120}\frac{\delta^7 y}{\delta x^7}h^4-\cdots \\
y''''(x) &= \frac{\delta^4 y}{\delta x^4}-\frac{1}{6}\frac{\delta^6 y}{\delta x^6}h^2+\frac{7}{240}\frac{\delta^8 y}{\delta x^8}h^4-\cdots
\end{aligned}
$$

前進差分

$$
\begin{aligned}
\frac{\varDelta y}{\varDelta x} &= \frac{1}{h}\{y(x+h)-y(x)\} \\
\frac{\varDelta^2 y}{\varDelta x^2} &= \frac{1}{h^2}\{y(x+2h)-2y(x+h)+y(x)\} \\
\frac{\varDelta^3 y}{\varDelta x^3} &= \frac{1}{h^3}\{y(x+3h)-3y(x+2h)+3y(x+h)-y(x)\} \\
\frac{\varDelta^4 y}{\varDelta x^4} &= \frac{1}{h^4}\{y(x+4h)-4y(x+3h)+6y(x+2h)-4y(x+h)+y(x)\} \\
\frac{\varDelta^5 y}{\varDelta x^5} &= \frac{1}{h^5}\{y(x+5h)-5y(x+4h)+10y(x+3h)-10y(x+2h) \\
&\qquad +5y(x+h)-y(x)\} \\
\frac{\varDelta^6 y}{\varDelta x^6} &= \frac{1}{h^6}\{y(x+6h)-6y(x+5h)+15y(x+4h)-20y(x+3h) \\
&\qquad +15y(x+2h)-6y(x+h)+y(x)\} \\
\frac{\varDelta^7 y}{\varDelta x^7} &= \frac{1}{h^7}\{y(x+7h)-7y(x+6h)+21y(x+5h)-35y(x+4h) \\
&\qquad +35y(x+3h)-21y(x+2h)+7y(x+h)-y(x)\} \\
\frac{\varDelta^8 y}{\varDelta x^8} &= \frac{1}{h^8}\{y(x+8h)-8y(x+7h)+28y(x+6h)-56y(x+5h) \\
&\qquad +70y(x+4h)-56y(x+3h)+28y(x+2h)-8y(x+h)+y(x)\}
\end{aligned}
$$

前進差分による補正公式

$$
\begin{aligned}
y'(x) &= \frac{\varDelta y}{\varDelta x}-\frac{1}{2}\frac{\varDelta^2 y}{\varDelta x^2}h+\frac{1}{3}\frac{\varDelta^3 y}{\varDelta x^3}h^2-\frac{1}{4}\frac{\varDelta^4 y}{\varDelta x^4}h^3+\frac{1}{5}\frac{\varDelta^5 y}{\varDelta x^5}h^4-\frac{1}{6}\frac{\varDelta^6 y}{\varDelta x^6}h^5 \\
&\qquad +\frac{1}{7}\frac{\varDelta^7 y}{\varDelta x^7}h^6-\frac{1}{8}\frac{\varDelta^8 y}{\varDelta x^8}h^7+\cdots \\
y''(x) &= \frac{\varDelta^2 y}{\varDelta x^2}-\frac{\varDelta^3 y}{\varDelta x^3}h+\frac{11}{12}\frac{\varDelta^4 y}{\varDelta x^4}h^2-\frac{5}{6}\frac{\varDelta^5 y}{\varDelta x^5}h^3+\frac{137}{180}\frac{\varDelta^6 y}{\varDelta x^6}h^4-\frac{7}{10}\frac{\varDelta^7 y}{\varDelta x^7}h^5 \\
&\qquad +\frac{363}{560}\frac{\varDelta^8 y}{\varDelta x^8}h^6-\cdots \\
y'''(x) &= \frac{\varDelta^3 y}{\varDelta x^3}-\frac{3}{2}\frac{\varDelta^4 y}{\varDelta x^4}h+\frac{7}{4}\frac{\varDelta^5 y}{\varDelta x^5}h^2-\frac{15}{8}\frac{\varDelta^6 y}{\varDelta x^6}h^3+\frac{29}{15}\frac{\varDelta^7 y}{\varDelta x^7}h^4 \\
&\qquad -\frac{469}{240}\frac{\varDelta^8 y}{\varDelta x^8}h^5+\cdots \\
y''''(x) &= \frac{\varDelta^4 y}{\varDelta x^4}-2\frac{\varDelta^5 y}{\varDelta x^5}h+\frac{17}{6}\frac{\varDelta^6 y}{\varDelta x^6}h^2-\frac{7}{2}\frac{\varDelta^7 y}{\varDelta x^7}h^3+\frac{967}{240}\frac{\varDelta^8 y}{\varDelta x^8}h^4-\cdots
\end{aligned}
$$

中心差分による補正公式を検証する Mathematica プログラム

```
(*
h = the step width
y0 = y(x), y1 = 1st derivative of y(x), y2 = 2nd derivative of y(x), ...
           d1 = 1st central difference, d2 = 2nd central difference, ...
*)

Clear[y0,y1,y2,y3,y4,y5,y6,y7,y8,d1,d2,d3,d4,d5,d6,d7,d8,h];

y[h_]:=y0+y1*h+y2*h^2/2+y3*h^3/6+y4*h^4/24+y5*h^5/120+y6*h^6/720+y7*h^7/5040+y8*h^8/40320;
(* Taylor expansion of y(x+h) *)

main[]:=Module[{},
  d1=Expand[(y[h]-y[-h])/(2*h)];
  d2=Expand[(y[h]-2*y[0]+y[-h])/(h^2)];
  d3=Expand[(y[2*h]-2*y[h]+2*y[-h]-y[-2*h])/(2*h^3)];
  d4=Expand[(y[2*h]-4*y[h]+6*y[0]-4*y[-h]+y[-2*h])/(h^4)];
  d5=Expand[(y[3*h]-4*y[2*h]+5*y[h]-5*y[-h]+4*y[-2*h]-y[-3*h])/(2*h^5)];
  d6=Expand[(y[3*h]-6*y[2*h]+15*y[h]-20*y[0]+15*y[-h]-6*y[-2*h]+y[-3*h])/(h^6)];
  d7=Expand[(y[4*h]-6*y[3*h]+14*y[2*h]-14*y[h]+14*y[-h]-14*y[-2*h]+6*y[-3*h]-y[-4*h])
             /(2*h^7)];
  d8=Expand[(y[4*h]-8*y[3*h]+28*y[2*h]-56*y[h]+70*y[0]-56*y[-h]+28*y[-2*h]-8*y[-3*h]
             +y[-4*h])/(h^8)];
  Print[];
  Print["d1 = ",InputForm[d1]];
  Print["d2 = ",InputForm[d2]];
  Print["d3 = ",InputForm[d3]];
  Print["d4 = ",InputForm[d4]];
  Print["d5 = ",InputForm[d5]];
  Print["d6 = ",InputForm[d6]];
  Print["d7 = ",InputForm[d7]];
  Print["d8 = ",InputForm[d8]];
  Print["----------"];
  z=Expand[d1-d3*h^2/6+d5*h^4/30-d7*h^6/140];
  Print["d1-d3*h^2/6+d5*h^4/30-d7*h^6/140 = ",InputForm[z]];
  z=Expand[d2-d4*h^2/12+d6*h^4/90-d8*h^6/560];
  Print["d2-d4*h^2/12+d6*h^4/90-d8*h^6/560 = ",InputForm[z]];
  z=Expand[d3-d5*h^2/4+(7/120)*d7*h^4];
  Print["d3-d5*h^2/4+(7/120)*d7*h^4 = ",InputForm[z]];
  z=Expand[d4-d6*h^2/6+(7/240)*d8*h^4];
  Print["d4-d6*h^2/6+(7/240)*d8*h^4 = ",InputForm[z]];
];

main[];

(*
d1 = y1 + (h^2*y3)/6 + (h^4*y5)/120 + (h^6*y7)/5040
d2 = y2 + (h^2*y4)/12 + (h^4*y6)/360 + (h^6*y8)/20160
d3 = y3 + (h^2*y5)/4 + (h^4*y7)/40
d4 = y4 + (h^2*y6)/6 + (h^4*y8)/80
d5 = y5 + (h^2*y7)/3
d6 = y6 + (h^2*y8)/4
d7 = y7
d8 = y8
----------
d1-d3*h^2/6+d5*h^4/30-d7*h^6/140 = y1
d2-d4*h^2/12+d6*h^4/90-d8*h^6/560 = y2
d3-d5*h^2/4+(7/120)*d7*h^4 = y3
d4-d6*h^2/6+(7/240)*d8*h^4 = y4
*)
```